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Eigenfunctions of Durrmeyer-type modifications of Meyer–König and Zeller operators

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Abstract

In this paper we consider a generalization of different variants of Durrmeyer-type modifications of Meyer–König and Zeller operators. Eigenfunctions in terms of linear combinations of the functions $f_m(x) = (1-x)^{-m}$ are computed explicitly. From this we also derive a quite simple representation in terms of the functions $h_m(x) = x^m/(1-x)^m$.

Keywords: Positive linear operators; Durrmeyer-type operators; Eigenfunctions

1. Introduction and definitions of the MKZD-type operators

Durrmeyer-type modifications M_n, $n \in \mathbb{N}$, of positive linear operators were first introduced for the classical Bernstein operators by Durrmeyer [4]. Their eigenfunctions which turned out to be independent of n were given by Derriennic [3].

There are several definitions for Durrmeyer-type modifications of the Meyer–König and Zeller operators in Refs. [1,2,6] which can be regarded as special cases of our general definition.

In what follows, the operators are always defined for functions f, such that the series on the right-hand side is convergent. This is the case e.g. for $f \in L_1[0,1]$ but also for i.e. the nonintegrable functions $f_m(x) = (1-x)^m$, $m \in \mathbb{N}$, m sufficiently small.

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For $x \in \mathbb{R}$, $l \in \mathbb{N}_0$, we will use the notation of rising and falling factorials, i.e.

$$x^{\bar{l}} = \prod_{v=0}^{l-1}(x+v), x^{\underline{l}} = \prod_{v=0}^{l-1}(x-v) \quad \text{for } l \in \mathbb{N}, \ x^{\bar{0}} = x^{\underline{0}} = 1.$$

Definition 1. For fixed $v \in \mathbb{Z}$, $\kappa \in \mathbb{R}$, let $n \in \mathbb{N}$, such that $n + v \in \mathbb{N}$. Then we define

$$(M_n^{v,\kappa} f)(x) = \sum_{k=0, k>\kappa-1}^{\infty} m_{n,k}(x)(n+v)\int_0^1 m_{n+v,k-\kappa}(t)(1-t)^{-2} f(t)\,dt,$$

where

$$m_{n,k}(x) = \frac{(k+1)^{\bar{n}}}{n!} x^k (1-x)^{n+1}, \quad x \in [0,1].$$

Remark 2. The above definition gives a unified approach to the different definitions of Durrmeyer-type modifications of the Meyer–König and Zeller operators. To be more precise we note that $M_n^{1,0}$ corresponds to the definition given by Abel and Gupta [1], $M_n^{2,0}$ to the definition by Chen [2] and $M_n^{0,2}$ to the definition by Guo [6]. The generalization was already given in [9] for the case $v \in \mathbb{N}_0$, $\kappa = 0$, which did not include the definition given by Guo.

By using the same transform as in [10] for the Baskakov and Meyer–König and Zeller operators, i.e. $\sigma(x) = \frac{x}{1-x}$, $f(t) = g(\sigma(t))$, we derive

$$(M_n^{v,\kappa} f)(x) = (B_{n+1}^{v,\kappa} g)(\sigma(x)), \tag{1}$$

where the general Durrmeyer-type modifications of the Baskakov operator are given by

$$(B_{n+1}^{v,\kappa} h)(x) = \sum_{k=0, k>\kappa-1}^{\infty} b_{n+1,k}(x)(n+v)\int_0^{\infty} b_{n+v+1,k-\kappa}(t) f(t)\,dt,$$

$$b_{n+1,k}(x) = \frac{(n+1)^{\bar{n}}}{n!} x^k (1+x)^{-(n+1+k)}, \quad x \in [0,\infty).$$

For $v = 0$ this is the Baskakov–Durrmeyer operator considered in several papers by the author (see e.g. [7,8]). This connection between the operators will be used at the end of the paper for the special case $v = \kappa = 0$.

In view of (1) $M_n^{0,0}$ seems to be the most natural case but up to the authors knowledge this has not been yet considered.

Throughout this paper we will use the following identities which can be easily verified:

$$\sum_{k=0}^{\infty} m_{n,k}(x) = 1, \tag{2}$$

$$\int_0^1 m_{n,k}(t)(1-t)^{-2}\,dt = \frac{1}{n}. \tag{3}$$

2. Eigenfunctions of the operators

In order to derive eigenfunctions of the operators $M_n^{v,\kappa}$ we have to find functions for which the image under the operators can be evaluated explicitly and for which the image is of the same type of function.

In case of $\kappa<1$ this leads us to the functions $f_m(t)=(1-t)^{-m}$, $m\in\mathbb{N}_0$. For $\kappa\geqslant 1$ this is not a suitable choice as it turns out that in this case the image of f_m is not of the same type (see Remark 4). Furthermore, it is easy to see that in case $\kappa\in\mathbb{N}$ the monomial $e_\kappa(t)=t^\kappa$ is an eigenfunction of $M_n^{v,\kappa}$ (see [6, p. 13] for $\kappa=2$). Unfortunately, this is the only one which we are able to determine up to now in this case.

Lemma 3. *Let $\kappa<1$, $f_m(t)=(1-t)^{-m}$, $m\in\mathbb{N}_0$ with $m\leqslant n+v-1$. Then we have*

$$(M_n^{v,\kappa}f_m)(x) = \frac{1}{(n+v-1)^{\underline{m}}n!}\sum_{j=0}^{m}\binom{m}{j}(n+j)!(v-\kappa-m)^{\overline{m-j}}f_j(x)$$

for $x\in[0,1)$.

Proof. With the definition of $m_{n+v,k-\kappa}(t)$ and (3) we get

$$\begin{aligned}
&(n+v)\int_0^1 m_{n+v,k-\kappa}(t)(1-t)^{-2}(1-t)^{-m}\,dt\\
&\quad=(n+v-m)\frac{(n+v+k-\kappa)^{\underline{m}}}{(n+v-1)^{\underline{m}}}\int_0^1 m_{n+v-m,k-\kappa}(t)(1-t)^{-2}\,dt\\
&\quad=\frac{(n+v+k-\kappa)^{\underline{m}}}{(n+v-1)^{\underline{m}}}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&(M_n^{v,\kappa}f_m)(x)\\
&\quad=\frac{1}{(n+v-1)^{\underline{m}}n!}\sum_{k=0}^{\infty}x^k(1-x)^{n+1}(k+1)^{\bar{n}}(n+v+k-\kappa)^{\underline{m}}.
\end{aligned} \tag{4}$$

We now determine coefficients $C_{m,j}$ such that

$$(k+1)^{\bar{n}}(n+v+k-\kappa)^{\underline{m}} = \sum_{j=0}^{m} C_{m,j}(k+1)^{\overline{n+j}}$$

which is equivalent to

$$(n+v+k-\kappa)^{\underline{m}} = \sum_{j=0}^{m} C_{m,j}(n+k+1)^{\bar{j}}.$$

This can be done by determining the Newton-polynomial of

$$p(x) = (x+v-\kappa)^{\underline{m}}$$

for the knots $x_l = -(1+l)$, $l = 0, 1, \dots, m$, i.e.

$$p(x) = \sum_{j=0}^{m} C_{m,j}(x+1)^{\bar{j}}$$

and evaluate this for $x = n+k$.

The corresponding divided differences are

$$C_{m,j} = \binom{m}{j}(v-\kappa-m)^{\overline{m-j}}, \quad j = 0, \dots, m.$$

Putting this into formula (4) we get after interchanging the order of summation which is allowed as the series is absolutely convergent for $x \in [0,1)$

$$\begin{aligned}(M_n^{v,\kappa} f_m)(x) \\ &= \frac{1}{(n+v-1)^{\underline{m}} n!} \sum_{j=0}^{m} \binom{m}{j}(n+j)!(v-\kappa-m)^{\overline{m-j}} f_j(x) \sum_{k=0}^{\infty} m_{n+j,k}(x) \\ &= \frac{1}{(n+v-1)^{\underline{m}} n!} \sum_{j=0}^{m} \binom{m}{j}(n+j)!(v-\kappa-m)^{\overline{m-j}} f_j(x), \qquad (5)\end{aligned}$$

where we used (2) in the last step. □

Remark 4. We point out that in case $m \geqslant v-\kappa > 0$, $-\kappa \in \mathbb{N}_0$, the rising factorials $(v-\kappa-m)^{\overline{m-j}}$ are equal zero for $j = 0, \dots, v-\kappa-1$. Thus for $m \geqslant v-\kappa > 0$ the result of Lemma 3 can also be rewritten into

$$(M_n^{v,\kappa} f_m)(x) = \frac{1}{(n+v-1)^{\underline{m}} n!} \sum_{j=v-\kappa}^{m} \binom{m}{j}(n+j)!(v-\kappa-m)^{\overline{m-j}} f_j(x).$$

In case of $\kappa \geqslant 1$ we get with the same calculations as above

$$\begin{aligned}(M_n^{v,\kappa} f_m)(x) \\ &= \frac{1}{(n+v-1)^{\underline{m}} n!} \sum_{j=0}^{m} \binom{m}{j}(n+j)!(v-\kappa-m)^{\overline{m-j}} f_j(x) \sum_{k=0, k>\kappa-1}^{\infty} m_{n+j,k}(x)\end{aligned}$$

instead of (5), from which it is clear that the choice f_m is not suitable for our considerations in this case.

With the result of Lemma 3 we are now able to prove the following statements about eigenfunctions and corresponding eigenvalues in case $\kappa < 1$.

Theorem 5. *Let $\kappa < 1$, $m \in \mathbb{N}_0$ with $m \leqslant n + \nu - 1$.*

(1) *For $m \geqslant \nu$ or $m \leqslant \frac{\nu-1}{2}$ the operator $M_n^{\nu,\kappa}$ has an eigenfunction of the form*

$$g_m^{\nu,\kappa} = \sum_{\mu=0}^{m} a_{m,\mu}^{\nu,\kappa} f_\mu \tag{6}$$

with corresponding eigenvalue

$$\lambda_{n,m}^{\nu} = \frac{(n+m)^{\underline{m}}}{(n+\nu-1)^{\underline{m}}}.$$

(2) *Except of a constant factor the coefficients of* (6) *can be uniquely determined by*

$$a_{m,\mu}^{\nu,\kappa} = (-1)^{m-\mu} \binom{m}{\mu} \frac{(m+\kappa-\nu)^{\underline{m-\mu}}}{(2m-\nu)^{\underline{m-\mu}}}. \tag{7}$$

(3) *For $\frac{\nu}{2} \leqslant m \leqslant \nu - 1$ there exists no linear combination of the functions f_μ, $\mu = 0, \ldots, m$, which is an eigenfunction of the operator $M_n^{\nu,\kappa}$.*

Proof. First, we investigate the equation

$$\left[M_n^{\nu,\kappa} \left(\sum_{\mu=0}^{m} a_{m,\mu}^{\nu,\kappa} f_\mu \right) \right](x) = \lambda_{n,m}^{\nu} \sum_{j=0}^{m} a_{m,j}^{\nu,\kappa} f_j(x). \tag{8}$$

With the result of Lemma 3 we get after interchanging the order of summation that (8) is equivalent to

$$\frac{1}{n!} \sum_{j=0}^{m} (n+j)! f_j(x) \sum_{\mu=j}^{m} \binom{\mu}{j} \frac{(\nu-\kappa-\mu)^{\overline{\mu-j}}}{(n+\nu-1)^{\underline{\mu}}} a_{m,\mu}^{\nu,\kappa} = \lambda_{n,m}^{\nu} \sum_{j=0}^{m} a_{m,j}^{\nu,\kappa} f_j(x). \tag{9}$$

Without loss of generality, we can assume $a_{m,m}^{\nu,\kappa} \neq 0$. Otherwise we could have considered $m-1$ instead of m. So in what follows we set $a_{m,m}^{\nu,\kappa} = 1$.

Comparing the coefficients of f_m leads to

$$\lambda_{n,m}^{\nu} = \frac{(n+m)^{\underline{m}}}{(n+\nu-1)^{\underline{m}}}.$$

Putting this in Eq. (9) multiplied by $n!(n+\nu-1)^{\underline{m}}$ and comparing the coefficients of f_j, $j = 0, \ldots, m-1$, leads to the following linear system of equations for the $a_{m,j}^{\nu,\kappa}$, $j = 0, \ldots, m-1$.

$$
\begin{aligned}
&a_{m,j}^{\nu,\kappa}\{(n+m)! - (n+j)!(n+\nu-j-1)^{\underline{m-j}}\}\\
&\quad - (n+j)!\sum_{\mu=j+1}^{m-1}\binom{\mu}{j}(n+\nu-\mu-1)^{\underline{m-\mu}}(\nu-\kappa-\mu)^{\overline{\mu-j}}a_{m,\mu}^{\nu,\kappa}\\
&= (n+j)!\binom{m}{j}(\nu-\kappa-m)^{\overline{m-j}}. \qquad (10)
\end{aligned}
$$

Since the coefficient matrix is triangular its determinant is given by the product of the diagonal entries, i.e.

$$
\begin{aligned}
&\prod_{j=0}^{m-1}\{(n+m)! - (n+\nu-j-1)^{\underline{m-j}}(n+j)!\}\\
&= \prod_{j=0}^{m-1}(n+j)!\left\{(n+m)^{\underline{m-j}} - (n+\nu-j-1)^{\underline{m-j}}\right\}.
\end{aligned}
$$

As $m<n+\nu$ by the assumption of the theorem we have in case $m \geqslant \nu$ that $n+j+1>n+\nu-m>0, n+j+2>n+\nu-m+1>0, \ldots, n+m>n+\nu-j-1>0$ which gives that the value of the determinant is different from zero. So system (10) has a unique solution which proves that there exists an eigenfunction of the desired form (6) with coefficients $a_{m,\mu}^{\nu,\kappa}$ which are unique except of a constant factor.

For $m \leqslant \frac{\nu-1}{2}$ it follows that $0<n+j+1<n+\nu-m$, $0<n+j+2<n+\nu-m+1, \ldots, 0<n+m<n+\nu-j-1$ which means that the value of the determinant is different from zero also in this case. So we also have eigenfunctions of the form (6) with coefficients $a_{m,\mu}^{\nu,\kappa}$ which are unique except of a constant factor in this case.

We now prove that for $m \geqslant \nu$ or $m \leqslant \frac{\nu-1}{2}$ the coefficients $a_{m,\mu}^{\nu,\kappa}$ in (7) satisfy (8) or equivalently that

$$
\begin{aligned}
&\sum_{\mu=j}^{m}(n+\nu-\mu-1)^{\underline{m-\mu}}\binom{\mu}{j}(\nu-\kappa-\mu)^{\overline{\mu-j}}(-1)^{m-\mu}\binom{m}{\mu}\frac{(m+\kappa-\nu)^{\underline{m-\mu}}}{(2m-\nu)^{\underline{m-\mu}}}\\
&= \frac{(n+m)!}{(n+j)!}(-1)^{m-j}\binom{m}{j}\frac{(m+\kappa-\nu)^{\underline{m-j}}}{(2m-\nu)^{\underline{m-j}}} \qquad (11)
\end{aligned}
$$

holds true for every $j = 0, \ldots, m$.

Multiplying by $(-1)^{m-j}\frac{j!}{m!}\frac{(2m-\nu)^{\underline{m-j}}}{(m+\kappa-\nu)^{\underline{m-j}}}$ and using the identities

$$
\begin{aligned}
&(-1)^{\mu-j}(\nu-\kappa-\mu)^{\overline{\mu-j}}\frac{(m+\kappa-\nu)^{\underline{m-\mu}}}{(m+\kappa-\nu)^{\underline{m-j}}} = 1,\\
&\frac{(2m-\nu)^{\underline{m-j}}}{(2m-\nu)^{\underline{m-\mu}}} = (m-\nu+\mu)^{\underline{\mu-j}},
\end{aligned}
$$

we get that (11) is equivalent to

$$\sum_{\mu=j}^{m} \frac{(n+\nu-\mu-1)^{\underline{m-\mu}}}{(m-\mu)!} \frac{(m-\nu+\mu)^{\underline{\mu-j}}}{(\mu-j)!} = \binom{n+m}{m-j}. \tag{12}$$

Shifting the index of summation $\tilde{\mu} = \mu - j$ we get that the left-hand side of (12) equals

$$\sum_{\tilde{\mu}=0}^{m-j} \frac{(n+\nu-\tilde{\mu}-j-1)^{\underline{m-\tilde{\mu}-j}}}{(m-j-\tilde{\mu})!} \frac{(m-\nu+\tilde{\mu}+j)^{\underline{\tilde{\mu}}}}{\tilde{\mu}!}.$$

By Gould [5, (3.2)] this is equal to $\binom{n+m}{m-j}$. So we have proved (11) and (8), respectively.

Together with our considerations above we have proved the first and second statement of our theorem.

In order to prove the third proposition we look at the case $\frac{\nu}{2} \leqslant m \leqslant \nu - 1$.

We consider again system (10) of linear equations for the coefficients $a_{m,j}^{\nu,\kappa}$ but only for $j = \nu - m, \ldots, m$. By the same arguments as above we get that

$$a_{m,j}^{\nu,\kappa} = (-1)^{m-j} \binom{m}{j} \frac{(m+\kappa-\nu)^{\underline{m-j}}}{(2m-\nu)^{\underline{m-j}}},$$

$j = \nu - m, \ldots, m$ is the unique solution except of a constant factor.

We now look at the equation for $j^* = \nu - m - 1$. Putting in the values for $a_{m,\mu}^{\nu,\kappa}$, $\mu = \nu - m, \ldots, m$ and $\lambda_{n,m}^{\nu}$ we get

$$\begin{aligned} & a_{m,j^*}^{\nu,\kappa} \{(n+m)! - (n+j^*)!(n+\nu-j^*-1)^{\underline{m-j^*}}\} \\ & \quad = (n+j^*)! \sum_{\mu=j^*+1}^{m} (n+\nu-\mu-1)^{\underline{m-\mu}} \binom{\mu}{j^*} \\ & \qquad \times (\nu-\kappa-\mu)^{\overline{\mu-j^*}} (-1)^{m-\mu} \binom{m}{\mu} \frac{(m+\kappa-\nu)^{\underline{m-\mu}}}{(2m-\nu)^{\underline{m-\mu}}}. \end{aligned} \tag{13}$$

Obviously, the value of the curly bracket on the left-hand side is equal to zero for $j^* = \nu - m - 1$.

As

$$(\nu-\kappa-\mu)^{\overline{\mu-\nu+m+1}} (-1)^{m-\mu} (m+\kappa-\nu)^{\underline{m-\mu}} = (m-\kappa)^{\underline{2m-\nu+1}},$$

we get for the right-hand side of (13)

$$\sum_{\mu=\nu-m}^{m} \binom{\mu}{\nu-m-1} \binom{m}{\mu} (n+\nu-\mu-1)! \frac{(m-\kappa)^{\underline{2m-\nu+1}}}{(2m-\nu)^{\underline{m-\mu}}}.$$

As every factor in this sum is positive it is different from zero. So Eq. (13) has no solution and we have proved the third proposition of our theorem. □

Remark 6. We want to remark that the eigenfunctions in Theorem 5 are independent of n. Furthermore, the corresponding eigenvalues do not depend on the parameter $\kappa < 1$.

Evidently, the third proposition of Theorem 5 is only relevant in case $v \geqslant 2$, as the inequality $\frac{v}{2} \leqslant m \leqslant v-1$ has no solution $m \in \mathbb{N}_0$ if $v \leqslant 1$.

We also point out that for $m \geqslant v - \kappa > 0$, $-\kappa \in \mathbb{N}_0$, the falling factorials $(m+\kappa-v)^{\underline{m-\mu}}$ are equal to zero for $\mu = 0, 1, \ldots, v-\kappa-1$ and thus the corresponding coefficients $a_{m,\mu}^{v,\kappa}$. So the eigenfunctions reduce to

$$g_m^{v,\kappa} = \sum_{\mu=v-\kappa}^{m} a_{m,\mu}^{v,\kappa} f_\mu$$

in this case.

Example 7. We list some examples of eigenfunctions:

$$g_0^{v,\kappa} = f_0,$$

$$v \leqslant 1 \text{ or } v \geqslant 3 : g_1^{v,\kappa} = f_1 - \frac{v-\kappa-1}{v-2} f_0,$$

$$v \leqslant 2 \text{ or } v \geqslant 5 : g_2^{v,\kappa} = f_2 - 2 \cdot \frac{v-\kappa-2}{v-4} f_1 + \frac{(v-\kappa-1)(v-\kappa-2)}{(v-3)(v-4)} f_0.$$

For $v - \kappa > 0$, $-\kappa \in \mathbb{N}_0$ we have

$$g_{v-\kappa}^{v,\kappa} = f_{v-\kappa},$$

$$g_{v-\kappa+1}^{v,\kappa} = f_{v-\kappa+1} - \frac{v-\kappa+1}{v-2\kappa+2} f_{v-\kappa},$$

$$g_{v-\kappa+2}^{v,\kappa} = f_{v-\kappa+2} - 2 \cdot \frac{v-\kappa+2}{v-2\kappa+4} f_{v-\kappa+1} + \frac{(v-\kappa+1)(v-\kappa+2)}{(v-2\kappa+3)(v-2\kappa+4)} f_{v-\kappa}.$$

In our next lemma we show an equivalent representation of the eigenfunctions of Theorem 5 in terms of the functions $h_j(x) = \left(\frac{x}{1-x}\right)^j$, $j \in \mathbb{N}_0$, which turns out to be quite simple.

Lemma 8. *Let* $h_j(x) = \left(\frac{x}{1-x}\right)^j$, $j \in \mathbb{N}_0$. *Then the functions* $g_m^{v,\kappa}$ *of Theorem* 5 *can be rewritten into*

$$g_m^{v,\kappa} = \sum_{j=0}^{m} \binom{m}{j} \frac{(m-\kappa)^{\underline{m-j}}}{(2m-v)^{\underline{m-j}}} h_j.$$

Proof. We start with the representation of the functions $g_m^{v,\kappa}$ given in Theorem 5. Putting in $f_\mu = \sum_{j=0}^{\mu} \binom{\mu}{j} h_j$ and interchanging the order of summation leads to

$$g_m^{v,\kappa} = \sum_{j=0}^{m} h_j \sum_{\mu=j}^{m} a_{m,\mu}^{v,\kappa} \binom{\mu}{j}. \tag{14}$$

Putting in the formulas for the coefficients $a_{m,\mu}^{\nu,\kappa}$ and using the identities

$$(-1)^{m-\mu}(m+\kappa-\nu)^{\underline{m-\mu}} = (\nu-\kappa-\mu-1)^{\underline{m-\mu}}$$

and

$$\frac{1}{(2m-\nu)^{\underline{m-\mu}}} = \frac{(m-\nu+\mu)^{\underline{\mu-j}}}{(2m-\nu)^{\underline{m-j}}},$$

we get for the inner sum of (14)

$$\sum_{\mu=j}^{m} a_{m,\mu}^{\nu,\kappa}\binom{\mu}{j} = \frac{m!}{j!(2m-\nu)^{\underline{m-j}}}\sum_{\mu=j}^{m}\frac{(m-\nu+\mu)^{\underline{\mu-j}}}{(\mu-j)!}\frac{(\nu-\kappa-\mu-1)^{\underline{m-\mu}}}{(m-\mu)!}. \tag{15}$$

Shifting the indices $\tilde{\mu} = \mu - j$ and using [5, (3.2)] we derive that (15) equals

$$\frac{m!}{j!(2m-\nu)^{\underline{m-j}}}\sum_{\tilde{\mu}=0}^{m-j}\frac{(m-\nu+j+\tilde{\mu})^{\underline{\tilde{\mu}}}}{\tilde{\mu}!}\frac{(\nu-\kappa-1-\tilde{\mu}-j)^{\underline{m-j-\tilde{\mu}}}}{(m-j-\tilde{\mu})!}$$
$$= \binom{m}{j}\frac{(m-\kappa)^{\underline{m-j}}}{(2m-\nu)^{\underline{m-j}}}.$$

Inserting this into (14) proves the proposition. □

In the following, we consider only the special case $\nu = \kappa = 0$ and use the connection to the Baskakov–Durrmeyer operators. In [8, Lemma 3.1] we have proved the following result.

Lemma 9. *Let* $\varphi(x) = \sqrt{x(1+x)}$ *and* f, $\varphi^{2m}f^{(m)}$ *m-times differentiable on the interval* $[0,\infty)$. *Then we have*

$$\frac{d^m}{dx^m}\{\varphi(x)^{2m}(B_{n+1}^{0,0}f(t))^{(m)}(x)\} = \{B_{n+1}^{0,0}(\varphi(t)^{2m}f^{(m)}(t))^{(m)}\}(x). \tag{16}$$

For the monomials it is proved in [8, Satz 4.1]:

Lemma 10. *Let* $e_m(t) = t^m$, $m \in \mathbb{N}_0$. *Then*

$$(B_{n+1}^{0,0}e_m)(x) = \frac{(n-m-1)!}{(n-1)!}\sum_{j=0}^{m}\frac{m!}{j!}\binom{m}{j}\frac{(n+j)!}{n!}x^j.$$

So putting $f(t) = \frac{1}{m!} e_m(t)$ in Eq. (16) we derive

$$\{B_{n+1}^{0,0}[(\varphi(t)^{2m})^{(m)}]\}(x) = \frac{(n+m)!(n-m-1)!}{n!(n-1)!}(\varphi(x)^{2m})^{(m)}.$$

From this we get the following result for the eigenfunctions of the Durrmeyer modifications of the Baskakov and the Meyer–König and Zeller operators respectively in terms of a Rodriguez-type formula.

Theorem 11. *The functions*

$$\tilde{g}_m(\sigma) = \frac{d^m}{d\sigma^m}(\sigma(1+\sigma))^m$$

are eigenfunctions of the Baskakov–Durrmeyer operators $B_{n+1}^{0,0}$ *with eigenvalues*

$$\lambda_{n,m}^0 = \frac{(n+m)^{\underline{m}}}{(n-1)^{\underline{m}}}.$$

With $\sigma(x) = \frac{x}{1-x}$ *we also have*

$$g_m^{0,0}(x) = \tilde{g}_m(\sigma(x)) = D^m\left(\frac{x}{(1-x)^2}\right)^m$$

are eigenfunctions of the operator $M_n^{0,0}$ *where the differential operator* D^m *is defined by*

$$(Df)(x) = \frac{f'(x)}{\sigma'(x)}, \quad (D^m f)(x) = D^{m-1}(Df)(x).$$

The representation of eigenfunctions in terms of Rodriguez-type formulas for the general case will be considered in a forthcoming paper.

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